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Approximation of the Distribution of the Supremum of a Centered Random Walk. Application to the Local Score

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Abstract. Let $(X_n)_{n\geq 0}$ be a real random walk starting at 0, with centered increments bounded by a constant K. The main result of this study is: $|\mathbb{P}(S_n/\sqrt{n} \geq x) - \mathbb{P}(\sigma \sup_{0\leq u\leq 1} B_u \geq x)| \leq C(n,K)\sqrt{\ln n/n}$, where $x \geq 0, \sigma^2$ is the variance of the increments, S_n is the supremum at time n of the random walk, $(B_u, u \geq 0)$ is a standard linear Brownian motion and $C(n,K)$ is an explicit constant. We also prove that in the previous inequality S_n can be replaced by the local score and $\sup_{0\leq u\leq 1} B_u$ by $\sup_{0\leq u\leq 1} |B_u|$.

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1. Introduction

Let $(\xi_i)_{i\geq 1}$ be a sequence of i.i.d. random variables, with zero mean and variance σ^2. We denote by $(X_n)_{n\geq 0}$ the associated random walk:

$$X_0 = 0, \quad X_n = \sum_{i=1}^{n} \xi_i, \quad n \geq 1. \tag{1}$$

(1) The well known central limit theorem (CLT) tells us that for every x in $\mathbb{R}$, $\lim_{n\to\infty} \mathbb{P}(X_n/\sigma\sqrt{n} \geq x) = \mathbb{P}(G \geq x)$ where G is a $\mathcal{N}(0,1)$-Gaussian random variable. In practice it is often important to estimate the rate of convergence. Loëve (Billingsley, 1968 and Loéve, 1979, p. 288) has proved

$$\left|\mathbb{P}\left(\frac{X_n}{\sigma\sqrt{n}} \geq x\right) - \mathbb{P}(G \geq x)\right| \leq \frac{C\mathbb{E}[|\xi_1|^3]}{\sqrt{n}}; \quad x \in \mathbb{R}, n \geq 1; \tag{2}$$

where C is a constant.

(2) Suppose now that we are interested in the asymptotic behavior of S_n, as n goes to infinity, $S_n = \max_{0\leq i\leq n} X_i$. The CLT is not sufficient, we need a functional convergence result (Donsker's theorem; Billingsley, 1968, p. 68), which implies

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n}{\sigma\sqrt{n}} \geq x\right) = \mathbb{P}\left(\sup_{0\leq u\leq 1} B_u \geq x\right); \quad x \geq 0, \tag{3}$$

where $(B_t, t \geq 0)$ is a standard one-dimensional Brownian motion started at 0.

Since $\sup_{0\leq u\leq 1} B_u$ and $|B_1|$ are identically distributed, the right hand-side of (3) can be easily computed.

A priori the rate of convergence of $\mathbb{P}(S_n/\sigma\sqrt{n} \geq x)$ to $\mathbb{P}(\sup_{0\leq u\leq 1} B_u \geq x)$ is unknown.

(3) In Daudin *et al*. (2000), motivated by biological considerations, we established a similar result to (3):

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{H_n}{\sigma\sqrt{n}} \geq x\right) = \mathbb{P}(B_1^* \geq x); \quad x \geq 0, \tag{4}$$

where $H_n = \max_{0\leq i\leq j}(X_j - X_i)$ and $B_1^* = \sup_{0\leq t\leq 1} |B_t|$. Recall that the density function of B_1^* can be expressed through series (cf. Borodin and Salminen, 1996, p. 146 and annex A in Daudin *et al*., 2000).

The analysis of genetic sequences requires a precise estimate of $\mathbb{P}(H_n/\sigma\sqrt{n} \geq x)$. However, the rate of decay of $n \to |\mathbb{P}(H_n/\sigma\sqrt{n} \geq x) - \mathbb{P}(B_1^* \geq x)|$ is unknown. Therefore its knowledge would be useful.

(4) The aim of this work is to give effective bounds to

$$\delta_n(S) = \left|\mathbb{P}\left(\frac{S_n}{\sigma\sqrt{n}} \geq x\right) - \mathbb{P}\left(\sup_{0\leq u\leq 1} B_u \geq x\right)\right|,$$

and to

$$\delta_n(H) = \left|\mathbb{P}\left(\frac{H_n}{\sigma\sqrt{n}} \geq x\right) - \mathbb{P}\left(\sup_{0\leq u\leq 1} |B_u| \geq x\right)\right|.$$

We prove (cf. Theorems 1 and 2) the following inequality:

$$\delta_n(Z) \leq C\sqrt{\frac{\ln n}{n}},$$

where $Z = S$ or H and C is a computable constant which only depends on the law of (ξ_i).

Let us detail the organization of the paper. In Section 2 we deal with the supremum of a centered random walk and then we adapt the analysis to handle the local score. In Section 3, we check the accuracy of previous bounds through numerical tests.

2. Approximation of the Distribution of the Supremum

(1) Let $(\xi_i)_{i\geq 1}$ be a sequence of i.i.d. bounded random variables with 0 mean. We set

$$X_0 = 0, \quad X_n = \sum_{i=1}^{n} \xi_i, \quad n \geq 1. \tag{5}$$

We denote by σ^2 the variance of ξ_i and we assume

$$|\xi_n| \leq K, \quad \forall n \geq 1. \tag{6}$$

The main idea of our approach is to embed the random walk $(X_n)_{n\geq 0}$ in a Brownian motion. The random walk $(X_n)_{n\geq 0}$ can be actually considered as a Brownian motion stopped at an increasing sequence of stopping times.

We recall below the scheme introduced by Skorokhod (1965) which represents the random walk $(X_n)_{n\geq 0}$ as $(B_{T_n}, n \geq 0)$, where $(B_t, t \geq 0)$ is a standard one-dimensional Brownian motion started at 0, and $(T_n)_{n\geq 0}$ is an increasing sequence of stopping times. This representation is the key to our approach.

(2) If μ is a probability measure on $\mathbb{R}$ centered and having a finite first moment (i.e., $\int_{\mathbb{R}} |x|\mu(dx) < +\infty$ and $\int_{\mathbb{R}} x\mu(dx) = 0$) we know (Azema, 1979, and Vallois, 1983) that there exists a stopping time T such that

$$\text{the law of } B_T \text{ is } \mu, \tag{7}$$

and

$$(B_{T\wedge t}, t \geq 0) \quad \text{is a uniformly integrable martingale.} \tag{8}$$

(8) tells us that T can be chosen not too large.

In fact if μ has a compact support included in $[-A, A]$, maximal inequality and (8) imply

$$T \leq T^*(A), \tag{9}$$

where $T^*(A) = \{t \geq 0, |B_t| \geq A\}$. Conversely (9) implies (8).

In our approach we only deal with random walk having bounded increments. Then we restrict ourself to probability measures with compact support, or Brownian stopping time verifying (9).

Let $\mathscr{P}_c$ be the set of probability measures on $\mathbb{R}$, with compact support and centered. We denote by $(U(\mu))_{\mu\in\mathscr{P}_c}$ a family of stopping times such that

$$B_{U(\mu)} \sim \mu, \quad \mathrm{Supp}(\mu) \subset [-K, K], \quad U(\mu) \leq T^*(K). \tag{10}$$

In particular, if μ belongs to $\mathscr{P}_c$, we have the useful identity

$$\mathbb{E}\Big[(B_{U(\mu)})^2\Big] = \mathbb{E}[U(\mu)] < +\infty. \tag{11}$$

We need a little bit more than (10), we assume $\mathscr{P}_c$ has the following scaling property:

$$U(\mu_c) \overset{(d)}{=} c^2 U(\mu), \quad \text{for any } c > 0, \tag{12}$$

where μ_c is the image of μ by $x \mapsto cx$.

The two families of stopping times defined by Azema and Yor (1979) and Vallois (1983) verify these properties. Let α_a be the function

$$\alpha_a(x) = \mathbb{E}\Big[T^*(a)^2\Big(\mathrm{e}^{xT^*(a)} - 1\Big)\Big], \quad a > 0, \ 0 \leq x \leq \frac{\pi^2}{8a}. \tag{13}$$

We are now able to state the main result of this section, concerning the asymptotic behavior of S_M, as M goes to infinity, where $S_k = \max_{0 \le i \le k} X_i$.

THEOREM 1 *Let* $M_0 \ge 2$ *and* $\sigma' = \sigma/K$.

a. There exists $x_*(M_0) \in [0, \pi^2/8]$ *such that*

$$\sqrt{\frac{\ln M_0}{M_0}} \le x_*(M_0)\sqrt{\alpha_1(x_*(M_0))}, \tag{14}$$

$$\sqrt{\frac{\ln M_0}{M_0}} \le \frac{\alpha_1(x_*(M_0))^{3/2}}{\sigma'2(5 - 2\sigma'4 + 3\alpha_1(x_*(M_0)))}. \tag{15}$$

b. For any $M \ge M_0$,

$$\left|\mathbb{P}\left(\frac{S_M}{\sqrt{M}} \ge x\right) - \mathbb{P}\left(\sup_{0 \le u \le 1} B_u \ge \frac{x}{\sigma}\right)\right| \le \sqrt{\frac{\ln M}{M}}\hat{C}(M), \tag{16}$$

where

$$\hat{C}(M) = \frac{2}{\sigma'\sqrt{2\pi}}\frac{1}{\sqrt{\ln M}} + \frac{2}{\sqrt{\ln M}} + \sqrt{\frac{2}{\pi}}\frac{e^{-1/2}}{\sigma'2}\sqrt{\frac{5}{3} - \sigma'4 + \alpha_1(x_*(M_0))}. \tag{17}$$

REMARK 1 *The function* α_1 *is known:*

$$\alpha_1(x) = \beta''(x) - \beta''(0) = \beta''(x) - \frac{5}{3}, \quad 0 \le x < \frac{\pi^2}{8}, \tag{18}$$

where $\beta(x) = \mathbb{E}[e^{xT^*(1)}] = 1/\cos(\sqrt{2x});\ x \in [0, \pi^2/8]$.

Therefore, we can find numerically $x_*(M_0)$ verifying (14) and (15) and $\hat{C}(M)$ is explicit (see Section 3). Replacing $(X_n)_{n \ge 0}$ by $(-X_n)_{n \ge 0}$ in Theorem 1 and using the symmetry of Brownian motion (namely $(-B_t)_{t \ge 0} \overset{(d)}{=} (B_t)_{t \ge 0}$), we are allowed to substitute $\min_{0 \le i \le M} X_i$ into S_M in (16). Our scheme developed for the maximum is rich enough to be applied to the local score $(H_n)_{n \ge 0}$. This process is defined by

$$H_n = \max_{0 \le i \le j \le n}(X_j - X_i) = \max_{0 \le j \le n}\left(X_j - \min_{0 \le i \le j} X_i\right). \tag{19}$$

The analog of Theorem 1 involving the local score is the following:

THEOREM 2 *Let* $M_0 \ge 2$ *and* $\sigma' = \sigma/K$. *For any* $M \ge M_0$,

$$\left|\mathbb{P}\left(\frac{H_M}{\sqrt{M}} \ge x\right) - \mathbb{P}\left(\sigma \sup_{0 \le u \le 1} |B_u| \ge x\right)\right| \le \bar{C}(M)\sqrt{\frac{\ln M}{M}}, \tag{20}$$

where

$$\bar{C}(M) = \frac{4}{\sigma'\sqrt{2\pi}}\frac{2}{\sqrt{\ln M}} + \frac{2}{\sqrt{\ln M}} + \sqrt{\frac{8}{\pi}}\frac{\mathrm{e}^{-1/2}}{\sigma'2}\sqrt{\frac{5}{3} - \sigma'4 + \alpha_1(x_*(M_0))}, \tag{21}$$

$x_*(M_0)$ being a positive number in $x_*(M_0) \in [0, \pi^2/8]$ verifying (14) and (15).

REMARK 2 *The cumulative distribution of* $\sup_{0 \le u \le 1} |B_u|$ *is known:*

$$\mathbb{P}\left(\sup_{0 \le u \le 1} |B_u| \le x\right) = \frac{2}{\pi}\sum_{k \in \mathbb{Z}} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2}{8x^2}\right), \quad x > 0.$$

(3) In the sequel M is a scale parameter, M being an integer larger than 1.

We presently give a representation of the random walk $(X_k)_{k \ge 0}$ in terms of Brownian motion path.

PROPOSITION 3 *There exists a sequence of stopping times $(T_n)_{n \ge 0}$, such that*

$$T_0 = 0, \quad T_k = \sum_{1 \le i \le k} T'_i, \tag{22}$$

and

$$(\sigma B_{T_k}, k \ge 0) \overset{(d)}{=} \left(\frac{X_k}{\sqrt{M}}, k \ge 0\right), \tag{23}$$

where $(T'_i)_{i \ge 1}$ are independent random variables, each T'_i belonging to $U(\nu)$, ν being the common distribution of $\xi/\sigma\sqrt{M}$. In particular:

$$B_{T'_i} \overset{(d)}{=} \frac{\xi_i}{\sigma\sqrt{M}}.$$

Proof: We set $T_1 = U(\nu)$. Property (10) implies that $B_{T_1} \overset{(d)}{=} X_1/\sigma\sqrt{M} = \xi_1/\sigma\sqrt{M}$.

We know that $(B'_t = B_{t+T_1} - B_{T_1}, t \ge 0)$ is a one-dimensional Brownian motion, independent of B_{T_1}. Let T'_2 be a stopping time $U'(\nu)$ (associated with ν and $(B'_t; t \ge 0)$) such that $B'_{T'_2} \overset{(d)}{=} \xi_2/\sigma\sqrt{M}$, and

$$T'_2 \le \left\{t \ge 0, |B'_t| \ge \frac{K}{\sigma\sqrt{M}}\right\}.$$

Iterating this procedure, we define by induction an increasing sequence of random times $(T_k, k \ge 0)$ such that:

$$T'_1 = T_1, \tag{24}$$

$$B_{T_k + T'_{k+1}} - B_{T_k} = B_{T_{k+1}} - B_{T_k} \overset{(d)}{=} \frac{1}{\sigma\sqrt{M}}\xi_{k+1}, \quad \forall k \ge 0, \tag{25}$$

where

$$T_0 = 0, \quad T_k = T'_1 + \cdots + T'_k, \quad k \ge 1. \tag{26}$$

T'_{k+1} is a stopping time with respect to the filtration generated by the Brownian motion $(B_{T_k+t} - B_{T_k}; t \ge 0)$. In particular

$$(B_{T_k} - B_{T_{k-1}}; k \geq 1) \stackrel{(d)}{=} \left(\frac{\xi_k}{\sigma\sqrt{M}}, k \geq 1\right). \tag{27}$$

■

In our study we are looking for properties of the law of $S_M = \max_{0 \leq i \leq M} X_i$. Obviously it depends only on the law of the whole process $(X_k)_{k \geq 0}$. Therefore, we can choose any realization of the random walk $(X_k)_{k \geq 0}$. In the sequel of the paper, according to Proposition 3, we take:

$$X_k = \sigma\sqrt{M} B_{T_k}, \quad \forall\ k \geq 1. \tag{28}$$

We use the strength of (28) to obtain first bounds to $\mathbb{P}\big(S_M/\sqrt{M} \geq x\big)$. The key point of our method is the following lemma:

LEMMA 4 *We have:*

$$\frac{1}{\sqrt{M}} S_k \leq \sigma \sup_{0 \leq u \leq T_k} B_u \leq \frac{1}{\sqrt{M}} S_k + \frac{K}{\sqrt{M}}, \quad \forall k \geq 1, \tag{29}$$

$$\mathbb{P}\left(\sup_{0 \leq u \leq 1} B_u \geq \frac{1}{\sigma\sqrt{1-\varepsilon}}\left(x + \frac{K}{\sqrt{M}}\right)\right) - \mathbb{P}(|T_M - 1| \geq \varepsilon) \leq \mathbb{P}\left(\frac{S_M}{\sqrt{M}} \geq x\right), \tag{30}$$

$$\mathbb{P}\left(\frac{S_M}{\sqrt{M}} \geq x\right) \leq \mathbb{P}\left(\sup_{0 \leq u \leq 1} B_u \geq \frac{x}{\sigma\sqrt{1+\varepsilon}}\right) + \mathbb{P}(|T_M - 1| \geq \varepsilon), \tag{31}$$

for any $x \geq 0$ and $\varepsilon > 0$.

Proof:

a. (28) implies (29).

b. Let $\varepsilon > 0$ and $x \geq 0$. The first inequality in (29) implies:

$$\mathbb{P}\left(\frac{S_M}{\sqrt{M}} \geq x\right) \leq \mathbb{P}\left(\sup_{0 \leq u \leq T_M} B_u \geq \frac{x}{\sigma}\right).$$

We decompose the probability in the right hand side as follows:

$$\begin{aligned}\mathbb{P}\left(\sup_{0 \leq u \leq T_M} B_u \geq \frac{x}{\sigma}\right) \leq\ & \mathbb{P}(|T_M - 1| \geq \varepsilon) \\ & + \mathbb{P}\left(T_M \leq 1 + \varepsilon, \sup_{0 \leq u \leq T_M} B_u \geq \frac{x}{\sigma}\right) \\ \leq\ & \mathbb{P}(|T_M - 1| \geq \varepsilon) + \mathbb{P}\left(\sup_{0 \leq u \leq 1+\varepsilon} B_u \geq \frac{x}{\sigma}\right).\end{aligned}$$

Since the Brownian motion $(B_t, t \geq 0)$ has the scaling property:

$$(B_{tc}, t \geq 0) \stackrel{(d)}{=} (\sqrt{c} B_t, t \geq 0)$$

for any $c > 0$,

$$\sup_{0 \le u \le c} B_u \overset{(d)}{=} \sqrt{c} \sup_{0 \le u \le 1} B_u.$$

This achieves the proof of (31).

c. (30) is a direct consequence of the following inclusions:

$$\left\{ \sup_{0 \le u \le 1-\varepsilon} B_u \ge \frac{x}{\sigma} + \frac{K}{\sigma\sqrt{M}}, |T_M - 1| \le \varepsilon \right\}$$
$$\subset \left\{ \sup_{0 \le u \le T_M} B_u \ge \frac{x}{\sigma} + \frac{K}{\sigma\sqrt{M}}, |T_M - 1| \le \varepsilon \right\}$$
$$\subset \left\{ \frac{S_M}{\sqrt{M}} \ge x, |T_M - 1| \le \varepsilon \right\} \subset \left\{ \frac{S_M}{\sqrt{M}} \ge x \right\}. \quad \blacksquare$$

We note that (26) and (12) imply that

$$\mathbb{E}(T_M) = M\mathbb{E}(T_1) = M\mathbb{E}(B_{T_1}^2) = \frac{M}{\sigma^2 M}\mathbb{E}(\xi_1^2) = 1.$$

Moreover, $T_M = T'_1 + \cdots + T'_M$, and $(T'_i)_{1 \le i \le M}$ are i.i.d., then the weak law of large numbers implies that T_M converges to 1, in probability, as M goes to infinity. Consequently $\lim_{M \to \infty} \mathbb{P}(|T_M - 1| \ge \varepsilon) = 0$.

Recall that our goal is to look for effective bounds for $\mathbb{P}(S_M/\sqrt{M} \ge x)$, x and M being given.

This leads us to take ε as a function of M in order to minimize $\mathbb{P}(|T_M - 1| \ge \varepsilon)$. This can be done through a large deviation technique, because the stopping time $T^*(A)$ admits some small exponential moments. Since for every probability measure μ with compact support in $[-K, K]$ we have $U(\mu) \le T^*(K)$, there exists $A(\mu) > 0$ such that:

$$\mathbb{E}[\exp\{\lambda U(\mu)\}] < +\infty \Leftrightarrow \lambda < A(\mu). \tag{32}$$

LEMMA 5 *Let $M \ge 1$ and $\varepsilon \ge 1$. We assume that μ is centred and has a compact support, recall that μ is the common law of (ξ_i). Then for any $\lambda_1 \in [0, A(\mu)]$, $\lambda_2 > 0$, we have:*

$$\mathbb{P}(T_M - 1 \ge \varepsilon) \le \exp\{-Mf_\varepsilon(\lambda_1)\}, \tag{33}$$

$$\mathbb{P}(T_M - 1 \le -\varepsilon) \le \exp\{-Mg_\varepsilon(\lambda_2)\}, \tag{34}$$

where

$$f_\varepsilon(x) = \sigma^2(1+\varepsilon)x - \ln(\mathbb{E}[\exp(xU(\mu))]), \quad x < A(\mu), \tag{35}$$

and

$$g_\varepsilon(x) = -\sigma^2(1-\varepsilon)x - \ln(\mathbb{E}[\exp(-xU(\mu))]), \quad x \ge 0. \tag{36}$$

Proof: The crucial identity is:

$$T_M = T'_1 + \cdots + T'_M.$$

Recall that $(T'_i)_{1 \le i \le M}$ are independent and distributed as $T'_1 = T_1$.

(1) Let $\lambda > 0$. Then, using Markov's inequality

$$\begin{aligned}\mathbb{P}(T_M \geq 1+\varepsilon) &= \mathbb{P}(\exp\{\lambda(T_1' + \cdots + T_M')\} \geq \exp\{\lambda(1+\varepsilon)\}) \\ &\leq e^{-\lambda(1+\varepsilon)}(\mathbb{E}[e^{\lambda T_1}])^M. \end{aligned} \tag{37}$$

T_1 is a stopping time associated with the distribution of $\xi_1/\sigma\sqrt{M}$, so

$$T_1 = U(\mu_c), \quad \text{where } c = \frac{1}{\sigma\sqrt{M}}.$$

Using the scaling property (12):

$$\mathbb{E}[e^{\lambda T_1}] = \mathbb{E}\left[\exp\left\{\frac{\lambda}{\sigma^2 M}U(\mu)\right\}\right].$$

Then

$$\mathbb{P}(T_M \geq 1+\varepsilon) \leq \exp\left\{-M\left(\frac{\lambda}{M}(1+\varepsilon) - \ln\left(\mathbb{E}\left[\exp\left\{\frac{\lambda}{\sigma^2 M}U(\mu)\right\}\right]\right)\right)\right\}.$$

(33) follows immediately.

(2) As for (34) it is sufficient to replace (37) by:

$$\mathbb{P}(T_M \leq 1-\varepsilon) = \mathbb{P}(\exp\{-\lambda(T_1' + \cdots + T_M')\} \geq \exp\{-\lambda(1-\varepsilon)\}).$$ ■

LEMMA 6 *We set*

$$\alpha_K(A') = \mathbb{E}\left[T^*(K)^2\left(e^{A'T^*(K)} - 1\right)\right], \quad 0 < A' < \frac{\pi^2}{8K^2}, \tag{38}$$

$$\rho = \mathbb{E}(U(\mu)^2) - \sigma^4, \tag{39}$$

$$c_1(\mu) = \frac{\sigma^4}{2(\rho + \alpha_K(A'))}. \tag{40}$$

Then for any ε *in* $[0, A'(\rho + \alpha_K(A'))/\sigma^2]$, *we have:*

$$\mathbb{P}(T_M - 1 \geq \varepsilon) \leq \exp(-M\varepsilon^2 c_1(\mu)). \tag{41}$$

Proof:

(1) According to Lemma 5, the determination of an upper bound for $\mathbb{P}(T_M - 1 \geq \varepsilon)$ leads us to study f_ε. In this proof, ε, μ, K and $A' < A(\mu)$ are fixed, then f (resp. α) stands for f_ε (resp. $\alpha_K(A')$) and $U(\mu)$ will be denoted by U.

(2) We set

$$F_1(x) = \exp\left\{\sigma^2 x + \left(\frac{\rho+\alpha}{2}\right)x^2\right\} - L(x), \quad x \leq A' \tag{42}$$

where

$$L(x) = \mathbb{E}(\exp\{xU\}). \tag{43}$$

By a straightforward calculation we obtain:

$$F_1(0) = 0, \quad F_1'(0) = 0, \tag{44}$$

$$F_1''(x) = \left[\rho + \alpha + (\sigma^2 + (\rho + \alpha)x)^2\right] \exp\left\{\sigma^2 x + \left(\frac{\rho + \alpha}{2}\right)x^2\right\} - \mathbb{E}\left(U^2 e^{xU}\right). \tag{45}$$

Since ρ, α and x are positive numbers,

$$F_1''(x) \geq \rho + \alpha + \sigma^4 - \mathbb{E}(U^2 e^{xU}).$$

We have

$$\mathbb{E}(U^2 e^{xU}) = \mathbb{E}(U^2) + \mathbb{E}(U^2(e^{xU} - 1)).$$

But $0 \leq x \leq A'$ and $U \leq T^*(K)$, then

$$\mathbb{E}(U^2 e^{xU}) \leq \mathbb{E}(U^2) + \alpha,$$
$$F_1''(x) \geq 0; \quad x \in [0, A'].$$

As a result, F is a convex function on $[0, A']$, (44) implies that

$$F_1(x) \geq 0; \quad \forall x \in [0, A']. \tag{46}$$

(3) Recall that

$$\mathbb{E}\left(e^{xT^*(K)}\right) = \frac{1}{\cos\,(K\sqrt{2x})}, \quad 0 < x < \frac{\pi^2}{8K^2}. \tag{47}$$

Since $U \leq T^*(K)$, then $A(\mu) \geq \pi^2/8K^2$.

(4) It is easy to check that (46) is equivalent to

$$f(x) \geq \sigma^2 \varepsilon x - \frac{\rho + \alpha}{2}x^2; \quad \forall x \in [0, A'].$$

The maximum to $x \mapsto \sigma^2 \varepsilon x - \rho + \alpha/2x^2$ is achieved at $x_*(M_0) = \sigma^2 \varepsilon/\rho + \alpha$ and is equal to $\sigma^4 \varepsilon^2/2(\rho + \alpha)$. Moreover

$$x_* \leq A' \Longleftrightarrow \varepsilon \leq \frac{\rho + \alpha}{\sigma^2}A'.$$

Consequently (33) directly implies (41). ■

LEMMA 7 *For any* ε *in* $[0, \alpha_K(A')(\rho + \alpha_K(A'))/\sigma^4(3\rho + 3\alpha_K(A') + \sigma^4)]$, *the following inequality holds:*

$$\mathbb{P}(T_M - 1 \leq -\varepsilon) \leq \exp\,(-M\varepsilon^2 c_1(\mu)), \tag{48}$$

$\rho, \alpha_K(A')$ and $c_1(\mu)$ *being defined in Lemma 6.*

Proof: As in the proof of previous lemma, we set $\alpha = \alpha_K(A')$ and $U = U(\mu)$.

(1) Let us introduce

$$F_2(x) = \exp\left\{-\sigma^2 x + \left(\frac{\rho + \alpha}{2}\right)x^2\right\} - L(-x); \quad x \geq 0, \tag{49}$$

L being defined by (43).

Taking the two first derivatives of F_2, we obtain

$$F_2(0) = 0, \qquad F_2'(0) = 0 \tag{50}$$

$$F_2''(x) = F_3(x) - \mathbb{E}\big(U^2 e^{-xU}\big), \tag{51}$$

with

$$F_3(x) = \Big[\rho + \alpha + (-\sigma^2 + (\rho+\alpha)x)^2\Big] \exp\Big\{-\sigma^2 x + \Big(\frac{\rho+\alpha}{2}\Big)\, x^2\Big\}.$$

Since $\mathbb{E}(U^2 e^{-xU}) \le \mathbb{E}(U^2)$, then

$$F_2''(x) \ge F_3(x) - \mathbb{E}(U^2). \tag{52}$$

(2) We claim that F_3 is a convex function. Taking the two first derivatives of F_3, we have

$$\begin{aligned} F_3''(x) = &\Big[3(\rho+\alpha)^2 + 6(\rho+\alpha)(-\sigma^2 + (\rho+\alpha)x)^2 \\ &+ (-\sigma^2 + (\rho+\alpha)x)^4\Big] \exp\Big\{-\ \sigma^2 x + \Big(\frac{\rho+\alpha}{2}\Big)\, x^2\Big\}. \end{aligned}$$

But $\rho + \alpha > 0$ then $F_3''(x) \ge 0$. As a result

$$F_3(x) \ge F_3(0) + xF_3'(0), \quad x \ge 0.$$

But

$$F_3(0) = \alpha + \mathbb{E}(U^2); \quad F_3'(0) = -\sigma^2\big(3\rho + 3\alpha + \sigma^4\big) < 0.$$

Then $F_3(x) \ge \mathbb{E}(U^2)$ as soon as

$$F_3(0) + xF_3'(x) \ge \mathbb{E}(U^2).$$

This condition is equivalent to $x \in [0, \beta_1]$, where

$$\beta_1 = \frac{\alpha}{\sigma^2(3\rho + 3\alpha + \sigma^4)}.$$

Finally, due to (50) and (52), F_2 is a positive convex function on $[0, \beta_1]$.

(3) It is easy to check:

$$F_2(x) \ge 0 \Longleftrightarrow g_\varepsilon(x) \ge \sigma^2 \varepsilon x - \Big(\frac{\rho+\alpha}{2}\Big)\, x^2,$$

g_ε being the function introduced in Lemma 5. We apply (34) with $\lambda_2 = x_*(M_0) = \sigma^2\varepsilon/\rho + \alpha$, we get:

$$\mathbb{P}(T_M - 1 \le -\varepsilon) \le \exp\{-M\ g_\varepsilon(x_*(M_0))\} = \exp\{-M\ \varepsilon^2 c_1(\mu)\}.$$

Moreover, $x_*(M_0) < \beta_1 \Longleftrightarrow \varepsilon < \alpha(\rho+\alpha)/\sigma^4(3\rho + 3\alpha + \sigma^4)$. ■

LEMMA 8 *For any* $0 < \varepsilon < 1/2$,

$$\mathbb{P}\left(\sup_{0 \le u \le 1} B_u \ge \frac{x}{\sigma\sqrt{1+\varepsilon}}\right) \le c\varepsilon + \mathbb{P}\left(\sup_{0 \le u \le 1} B_u \ge \frac{x}{\sigma}\right), \tag{53}$$

where

$$c = \frac{1}{2}\sqrt{\frac{3}{\pi}}\,\mathrm{e}^{-1/2}.$$

Proof: As $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{0 \le u \le 1} B_u \ge \frac{x}{\sigma\sqrt{1+\varepsilon}}\right) = \mathbb{P}\left(\sup_{0 \le u \le 1} B_u \ge \frac{x}{\sigma}\right) + \delta,$$

where

$$\delta = \mathbb{P}\left(\frac{x}{\sigma\sqrt{1+\varepsilon}} \le \sup_{0 \le u \le 1} B_u \le \frac{x}{\sigma}\right).$$

But it is well known that $\sup_{0 \le u \le 1} B_u \overset{(d)}{=} |B_1|$, so that

$$\delta = \mathbb{P}\left(\frac{x}{\sigma\sqrt{1+\varepsilon}} \le |B_1| \le \frac{x}{\sigma}\right) = 2\mathbb{P}\left(\frac{x}{\sigma\sqrt{1+\varepsilon}} \le B_1 \le \frac{x}{\sigma}\right)$$
$$= 2\left(\Phi\left(\frac{x}{\sigma}\right) - \Phi\left(\frac{x}{\sigma\sqrt{1+\varepsilon}}\right)\right),$$

with

$$\Phi(z) = \mathbb{P}(B_1 \le z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \mathrm{e}^{-u^2/2} du.$$

Using a formula of finite increments, we obtain

$$\delta = 2\left(\frac{x}{\sigma} - \frac{x}{\sigma\sqrt{1+\varepsilon}}\right)\Phi'(y), \quad \text{for some } y \in \left[\frac{x}{\sigma\sqrt{1+\varepsilon}}; \frac{x}{\sigma}\right].$$

However,

$$0 < \frac{x}{\sigma} - \frac{x}{\sigma\sqrt{1+\varepsilon}} = \frac{x\varepsilon}{\sigma\sqrt{1+\varepsilon}\,(\sqrt{1+\varepsilon}+1)} \le \frac{x\varepsilon}{2\sigma}.$$

Suppose that $\varepsilon < 1/2$ and $y \in \left[x/\sigma\sqrt{1+\varepsilon}, x/\sigma\right]$, then

$$\Phi'(y) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-y^2/2} \le \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/(3\sigma^2)}.$$

So that

$$\delta \le \varepsilon h_0\left(\frac{x}{\sigma}\right),$$

where

$$h_0(z) = \frac{z}{\sqrt{2\pi}} \mathrm{e}^{-z^2/3}.$$

But $h_0(z) \leq h_0(\sqrt{3/2}) = c$, this shows (53). ∎

At this stage we have to give a lower bound to $\mathbb{P}(\sup_{0 \leq u \leq 1} B_u \geq 1/\sigma\sqrt{1-\varepsilon}\,(x + K/\sigma\sqrt{M}))$. Using same tools as for Lemma 8, we will prove:

LEMMA 9 *For any* $0 < \varepsilon < 1/2$,

$$\mathbb{P}\left(\sup_{0 \leq u \leq 1} B_u \geq \frac{x}{\sigma}\right) - \frac{2K}{\sigma\sqrt{2\pi M}} - c_2\,\varepsilon \leq \mathbb{P}\left(\sup_{0 \leq u \leq 1} B_u \geq \frac{1}{\sigma\sqrt{1-\varepsilon}}\left(x + \frac{K}{\sqrt{M}}\right)\right), \tag{54}$$

where

$$c_2 = \frac{2\mathrm{e}^{-1/2}}{\sqrt{2\pi}}.$$

Proof:

(1) We set $y = x + K/\sqrt{M}$. Using the same arguments as for Lemma 8, we obtain:

$$\mathbb{P}\left(\sup_{0 \leq u \leq 1} B_u \geq \frac{y}{\sigma\sqrt{1-\varepsilon}}\right) = \mathbb{P}\left(\sup_{0 \leq u \leq 1} B_u \geq \frac{y}{\sigma}\right) - \delta,$$

where

$$\delta = \mathbb{P}\left(\frac{y}{\sigma} \leq |B_1| \leq \frac{y}{\sigma\sqrt{1-\varepsilon}}\right) = 2\left(\Phi\left(\frac{y}{\sigma\sqrt{1-\varepsilon}}\right) - \Phi\left(\frac{y}{\sigma}\right)\right).$$

We have successively:

$$\delta = 2\left(\frac{y}{\sigma\sqrt{1-\varepsilon}} - \frac{y}{\sigma}\right)\Phi'(z), \quad \text{for some } z \in \left[\frac{y}{\sigma}; \frac{y}{\sigma\sqrt{1-\varepsilon}}\right].$$

Since $z \geq y/\sigma$,

$$\Phi'(z) \leq \frac{1}{\sqrt{2\pi}} \exp -\frac{y^2}{2\sigma^2},$$

and

$$\frac{y}{\sigma\sqrt{1-\varepsilon}} - \frac{y}{\sigma} = \frac{y}{\sigma}\left(\frac{1-\sqrt{1-\varepsilon}}{\sqrt{1-\varepsilon}}\right) = \frac{y}{\sigma}\left(\frac{\varepsilon}{(1+\sqrt{1-\varepsilon})(\sqrt{1-\varepsilon})}\right).$$

$$\delta \leq \varepsilon \frac{1}{(1+\sqrt{1-\varepsilon})(\sqrt{1-\varepsilon})} h_1\left(\frac{y}{\sigma}\right), \quad \text{with } h_1(z) = \frac{2z}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}.$$

but $\varepsilon \leq 1/2$, so that $\sqrt{1-\varepsilon} \geq 1/\sqrt{2}$, then

$$(1+\sqrt{1-\varepsilon})(\sqrt{1-\varepsilon}) \geq \frac{1}{\sqrt{2}}\left(1+\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}+1}{2} \geq 1.$$

We get

$$\delta \leq \varepsilon h_1\left(\frac{y}{\sigma}\right) \leq \varepsilon h_1(1) = \varepsilon c_2.$$

(2) We have to express $\mathbb{P}(\sup_{0\leq u\leq 1} B_u \geq y/\sigma)$ through $\mathbb{P}(\sup_{0\leq u\leq 1} B_u \geq x/\sigma)$.

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq u\leq 1} B_u \geq x/\sigma\right) - \mathbb{P}\left(\sup_{0\leq u\leq 1} B_u \geq \frac{y}{\sigma}\right)\\
&= \mathbb{P}\left(\frac{x}{\sigma} \leq \sup_{0\leq u\leq 1} B_u \leq \frac{x}{\sigma} + \frac{K}{(\sigma\sqrt{M})}\right)\\
&= 2\left(\Phi\left(\frac{x}{\sigma}+\frac{K}{\sigma\sqrt{M}}\right) - \Phi\left(\frac{x}{\sigma}\right)\right)\\
&\leq \frac{2K}{\sigma\sqrt{2\pi}\, M}\,\mathrm{e}^{-x^2/(2\sigma^2)}\\
&\leq \frac{2K}{\sigma\sqrt{2\pi}\, M}.
\end{aligned}$$

This ends the proof. ■

We are now able to prove Theorem 1. We can control the rate of convergence of the two probability distributions functions.

Proof of Theorem 1:

(1) Using Lemmas 4, 5–9, we obtain:

$$\begin{aligned}
&\left|\mathbb{P}\left(\frac{S_M}{\sqrt{M}} \geq x\right) - \mathbb{P}\left(\sup_{0\leq u\leq 1} B_u \geq \frac{x}{\sigma}\right)\right|\\
&\leq \max\left\{\frac{2K}{\sigma\sqrt{2\pi}\, M} + \frac{2\mathrm{e}^{-1/2}}{\sqrt{2\pi}}\varepsilon + 2\exp\{-c_1(\mu) M\varepsilon^2\},\right.\\
&\left.\frac{1}{2}\sqrt{\frac{3}{\pi}}\,\mathrm{e}^{-1/2}\varepsilon + 2\exp\{-c_1(\mu) M\varepsilon^2\}\right\}.
\end{aligned}$$

as soon as

$$\varepsilon \leq \frac{\rho+\alpha_K(A')}{\sigma^2}\min\left\{A',\ \frac{\alpha_K(A')}{\sigma^2(3\rho+3\alpha_K(A')+\sigma^4)}\right\}. \tag{55}$$

But

$$\frac{1}{2}\sqrt{\frac{3}{\pi}}\,\mathrm{e}^{-1/2} < \frac{2\,\mathrm{e}^{-1/2}}{\sqrt{2\pi}},$$

then

$$\left|\mathbb{P}\left(\frac{S_M}{\sqrt{M}} \geq x\right) - \mathbb{P}\left(\sup_{0 \leq u \leq 1} B_u \geq \frac{x}{\sigma}\right)\right|$$
$$\leq \frac{2K}{\sigma\sqrt{2\pi\, M}} + \frac{2\,\mathrm{e}^{-1/2}}{\sqrt{2\pi}}\varepsilon + 2\exp\{-c_1(\mu)M\varepsilon^2\}. \tag{56}$$

Minimizing in ε in the right hand side of (56) leads to $\varepsilon = \sqrt{\ln M/2Mc_1(\mu)}$. Hence

$$\left|\mathbb{P}\left(\frac{S_M}{\sqrt{M}} \geq x\right) - \mathbb{P}\left(\sup_{0 \leq u \leq 1} B_u \geq \frac{x}{\sigma}\right)\right|$$
$$\leq \sqrt{\frac{\ln M}{M}}\left(\frac{2}{\sqrt{\ln M}} + \frac{2K}{\sigma\sqrt{2\pi}}\frac{1}{\sqrt{\ln M}} + \sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{-1/2}}{\sigma^2}\sqrt{\rho + \alpha_K(A')}\right)$$

and (55) is equivalent to:

$$\sqrt{\frac{\ln M}{M}} \leq \sqrt{\rho + \alpha_K(A')}\min\left\{A', \frac{\alpha_K(A')}{\sigma^2(3\rho + 3\alpha_K(A') + \sigma^4)}\right\}. \tag{57}$$

(2) Using the scaling property of the Brownian motion (namely $T^*(K) \stackrel{(d)}{=} K^2T^*(1)$), we have:

$$\alpha_K(A') = K^4\alpha_1(A'K^2). \tag{58}$$

Let β be the function:

$$\beta(x) = \mathbb{E}\left[\mathrm{e}^{xT^*(1)}\right], \quad 0 \leq x < \frac{\pi^2}{8}, \tag{59}$$

then

$$\beta(x) = \frac{1}{\cos\,(\sqrt{2x})}, \quad 0 \leq x < \frac{\pi^2}{8}, \tag{60}$$
$$\alpha_1(x) = \beta''(x) - \beta''(0). \tag{61}$$

This allows us to compute explicitly α. It is clear that

$$\lim_{x \to \pi^2/8} \alpha_1(x) = +\infty, \tag{62}$$

and α_1 is an increasing function starting at 0.

Let M_0 be a fixed integer, $M_0 \geq 2$. Relation (61) implies that exists $x_1 \in [0, \pi^2/8]$ such that

$$x\sqrt{\alpha_1(x)} \geq \sqrt{\frac{\ln M_0}{M_0}}, \quad \forall x \in [x_1, \pi^2/8]. \tag{63}$$

Consequently the scaling property (58) yields to:

$$A'\sqrt{\rho + \alpha_K(A')} \geq \sqrt{\frac{\ln M_0}{M_0}}, \quad \forall A' \in [A_1', \pi^2/8K^2], \tag{64}$$

where $A_1' = x_1/K^2$.

(3) Let us determine an upper bound for $\mathbb{E}(U(\mu)^2)$.
Since $U(\mu) \leq T^*(K)$, then

$$\mathbb{E}(U(\mu)^2) \leq \mathbb{E}\ (T^*(K)^2) \leq K^4\mathbb{E}(T^*(1)^2) = \frac{5}{3}K^4. \tag{65}$$

As a result $\sqrt{\rho + \alpha_K(A')} \leq \sqrt{5/3K^4 - \sigma^4 + \alpha_K(A')}$.

(4) Using once more (62), we can find $x_2 \in [0, \pi^2/8]$ such that:

$$\frac{\alpha_1(x)^{3/2}}{\sigma'2(5 - 2\sigma'4 + 3\alpha_1(x))} \geq \sqrt{\frac{\ln M_0}{M_0}}, \tag{66}$$

for any $x \in [x_2, \pi^2/8]$, where $\sigma' = \sigma/K$.
As a result,

$$\frac{\alpha_K(A')\sqrt{\rho + \alpha_K(A')}}{\sigma^2(3\rho + 3\alpha_K(A') + \sigma^4)} \geq \sqrt{\frac{\ln M_0}{M_0}}, \tag{67}$$

for any $A' \in [x_2/K^2, \pi^2/8K^2]$.
It turns out that (57) holds for any $M \geq M_0$. ■

REMARK 3 *We have actually proved the existence of $M(\mu)$ such that for any $M \geq M(\mu)$:*

$$\left|\mathbb{P}\left(\frac{S_M}{\sqrt{M}} \geq x\right) - \mathbb{P}\left(\sup_{0\leq u\leq 1} B_u \geq \frac{x}{\sigma}\right)\right| \leq C(M,\mu)\sqrt{\frac{\ln M}{M}}, \tag{68}$$

where μ is the common distribution of ξ_i and

$$C(M,\mu) = \frac{2K}{\sigma\sqrt{2\pi}}\frac{1}{\sqrt{\ln M}} + \frac{2}{\sqrt{\ln M}} + \sqrt{\frac{2}{\pi}}\frac{e^{-1/2}}{\sigma^2}\sqrt{\mathbb{E}(U(\mu)^2) - \sigma^4 + \alpha_K(A')}. \tag{69}$$

and A' verifies (57).
A priori $C(M,\mu)$ is the best constant given by our approach, however, $M(\mu)$ has the disadvantage of not being explicit. This explains the formulation of Theorem 1.

Proof of Theorem 2: The method is the same as the one developed for the maximum. However, there are two changes.

a. (29) has to be replaced by:

$$\frac{1}{\sqrt{M}}H_k \leq \sigma \max_{0\leq u\leq T_k}\left(B_u - \min_{0\leq v\leq u} B_v\right) \leq \frac{1}{\sqrt{M}}H_k + \frac{2K}{\sqrt{M}}.$$

b. We need an upper-bound for $\mathbb{P}(a < \zeta < b)$, where $0 < a < b$ and ζ is the random variable

$$\zeta = \max_{0\leq u\leq 1}\left(B_u - \min_{0\leq v\leq u} B_v\right).$$

Recall that Lévy's theorem implies that $\zeta \overset{(d)}{=} B_1^*$, where $B_1^* = \sup_{0 \le u \le 1} |B_u|$.
If we set $S_1 = \sup_{0 \le u \le 1} B_u$ and $I_1 = \min_{0 \le u \le 1} B_u$, then

$$S_1 \overset{(d)}{=} I_1 \overset{(d)}{=} |B_1|$$

and

$$\{a < B_1^* < b\} \subset \{a < S_1 < b\} \cup \{a < -I_1 < b\},$$
$$\mathbb{P}(a < B_1^* < b) \le 2\mathbb{P}(a < |B_1| < b).$$

The rest of the proof runs as in Theorem 1. ■

3. Numerical Tests

This section is devoted to the numerical validation of our results: we would like to verify the quality of our upper bound $\hat{C}(M)$ (resp. $\bar{C}(M)$) in (17) (resp. (21)).

3.1. Three Classes of Examples of μ

For simplicity we consider only discrete probability measures. Let us recall that μ is the common distribution of ξ_i. We examine three classes $\mathscr{C}_1, \mathscr{C}_2$ and $\mathscr{C}_3$ of μ.

- $\mathscr{C}_1$ is constituted by uniform distributions on $\{-2, 2\}, \{-5, \dots, -1, 1, \dots, 5\}$ and $\{-10, \dots, -1, 1, \dots, 10\}$, noted $\mu_{1,1}, \mu_{1,2}$ and $\mu_{1,3}$ respectively.
- In $\mathscr{C}_2$ the three probability measures $\mu_{2,1}, \mu_{2,2}$ and $\mu_{2,3}$ are rather concentrated at the end points of their support. More precisely we choose:

$$\mu_{2,1} = \frac{1}{6} \sum_{i=-2, i \ne 0}^{2} |i| \delta_i,$$

$$\mu_{2,2} = \frac{1}{30} \sum_{i=-5, i \ne 0}^{5} |i| \delta_i,$$

$$\mu_{2,3} = \frac{1}{110} \sum_{i=-10, i \ne 0}^{10} |i| \delta_i,$$

  where δ_i denotes the Dirac measure at i.
- In $\mathscr{C}_3$ we consider $\mu_{3,1}, \mu_{3,2}$ and $\mu_{3,3}$ which are rather concentrated at the origin. We take:

$$\mu_{3,1} = \frac{1}{6} \sum_{i=-2, i\neq 0}^{2} (3 - |i|)\delta_i,$$

$$\mu_{3,2} = \frac{1}{30} \sum_{i=-5, i\neq 0}^{5} (6 - |i|)\delta_i,$$

$$\mu_{3,3} = \frac{1}{110} \sum_{i=-10, i\neq 0}^{10} (11 - |i|)\delta_i.$$

We observe that $K = 2$ (resp. $K = 5, K = 10$) for $\mu_{i,1}$ (resp. $\mu_{i,2}, \mu_{i,3}$), $1 \leq i \leq 3$.

3.2. The Supremum of a Random Walk

Let us explain our numerical procedure. We use the random number generator of the GSL library under GNU General Public License.

Let us start with M fixed. We generate k times the random walk $(X_i)_{0\leq i\leq M}$ and then obtain a k-sample of $S_M/\sqrt{M}$ whose empirical distribution function is denoted $F_{k,M}$. On one hand, Theorem 1 tells us

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{S_M}{\sqrt{M}} \leq x \right) - F\left(\frac{x}{\sigma} \right) \right| \leq \hat{C}(M) \sqrt{\frac{\ln M}{M}}, \tag{70}$$

where $F(x) = \mathbb{P}(|B_1| \leq x)$ and

$$\hat{C}(M) = \frac{2}{\sigma'\sqrt{2\pi}} \frac{1}{\sqrt{\ln M}} + \frac{2}{\sqrt{\ln M}} + \sqrt{\frac{2}{\pi}} \frac{\mathrm{e}^{-1/2}}{\sigma'2} \sqrt{\frac{5}{3} - \sigma'4 + \alpha_1(x_*(M_0))}. \tag{71}$$

On the other hand,

$$\sup_{x\in\mathbb{R}} \left| F_{k,M}(x) - F\left(\frac{x}{\sigma} \right) \right| \leq \delta_{k,M} \sqrt{\frac{\ln M}{M}} \tag{72}$$

with

$$\delta_{k,M} = \sqrt{\frac{M}{\ln M}} \left(\sup_{x\in\mathbb{R}} \left| F_{k,M}(x) - F\left(\frac{x}{\sigma} \right) \right| \right). \tag{73}$$

Kolmogorov's theorem implies that $\mathbb{P}(S_M/\sqrt{M} \leq x)$ can be approximated by $F_{k,M}(x)$, uniformly with respect to x, with heuristic rate $1/\sqrt{k}$. We choose $k = 10^6$.

This brings us to compare $\hat{C}(M)$ and $\delta_{k,M}$. We introduce

$$R(M) = \frac{\hat{C}(M)}{\delta_{k,M}}. \tag{74}$$

Then $R(M)$ close to 1 (resp. large) means that our upper bound $\hat{C}(M)$ is convenient (resp. over-estimated).

Recall that x_* depends on M_0 (cf. (14) and (15)). Then there are two ways to choose x_*.

1. The first length of random walk we consider is ten. So we fix $M_0 = 10$ and determine the corresponding value of $x_*(10)$ ($\hat{C}(M)$ being the constant given by (17), for any $M \geq M_0$).

This procedure is denoted by **F** on the legends of the graphs.

2. The simulations are led with M varying from 10 to 10,000, with step 10. We try to improve our procedure. For any M we determine the best value of $x_*(M)$ verifying (14) and (15). We choose $\hat{C}(M)$ by the relation (17) where $x_*(M_0)$ is replaced by $x_*(M)$.

We summarize the results in Table 1. For each kind of distribution, we write the minimum and the maximum over M, for $\delta_{k,M}$ and $R(M)$. The letter F (resp. V) recalls that we compute $\hat{C}(M)$ using $x_*(10)$ (resp. $x_*(M)$).

We also plot the two graphs of $M \mapsto R(M)$, from $M = 10$ to $M = 10{,}000$, corresponding to the F and the V procedures. We restrict ourself to $K = 5$, for the other cases, the graphs are similar. The obtained graphs are drawn in Figures 1 to 3.

We observe two facts: $R(M)$ seems to be constant if M is large enough. The ratio $R(M)$ is substantially lower with the V-procedure.

Table 1. Error factor.

			$\delta_{k,M}$		$R(M)$	
Class	K	Mod	Min	Max	Min	Max
1	2	F	0.21	0.44	6.4	12.9
		V	0.22	0.44	5.6	6.4
	5	F	0.21	0.40	8.9	17.2
		V	0.21	0.40	8.0	9.1
	10	F	0.19	0.38	10.2	20.5
		V	0.19	0.38	9.8	10.8
2	2	F	0.22	0.44	5.7	11.3
		V	0.22	0.44	4.8	5.7
	5	F	0.20	0.40	7.2	14.4
		V	0.20	0.40	6.7	7.2
	10	F	0.19	0.38	8.0	15.9
		V	0.19	0.38	7.5	8.0
3	2	F	0.23	0.45	7.2	13.9
		V	0.23	0.45	6.8	7.2
	5	F	0.20	0.40	7.2	14.4
		V	0.20	0.40	12.2	13.5
	10	F	0.19	0.39	15.5	31.0
		V	0.19	0.39	9.7	10.8

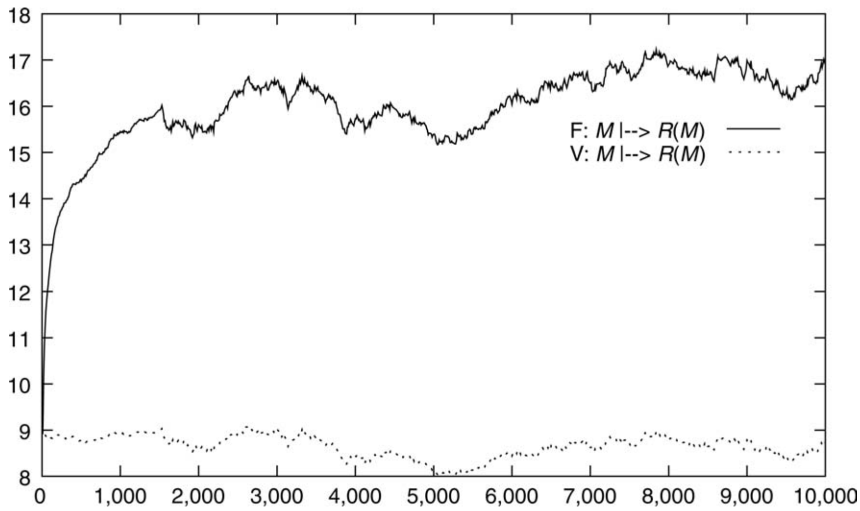

Figure 1. Graphs of $M \mapsto R(M)$, when $\mu = \mu_{1,2}$.

3.3. The Local Score

We use the same procedure to obtain the empirical cumulative distribution of the local score. We keep the same notations, i.e.,

$$\delta_{k,M} = \sqrt{\frac{M}{\ln M}} \left(\sup_{x \in \mathbb{R}} \left| F^{(S)}_{k,M}(x) - \mathbb{P}\left(\sigma \sup_{0 \le u \le 1} |B_u| \le x \right) \right| \right). \tag{75}$$

where $F^{(S)}$ is the empirical cumulative distribution of the local score divide by the square root of n.

To compare and $\bar{C}(M)$ $\delta_{k,M}$, we introduce:

$$\bar{R}(M) = \frac{\bar{C}(M)}{\delta_{k,M}}. \tag{76}$$

The number k of simulations is fixed to 10^6. We compute $\bar{C}(M)$ distinguishing the F and the V-method. We observe two facts: $R(M)$ seems to be constant if M is large enough, and the ratio $R(M)$ is substantially lower with the V-procedure.

The results are given in Table 2.

3.4. Conclusions About Numerical Results

The simulations show that our upper bound $C(M)$ (resp. $\bar{C}(M)$) for the supremum (resp. the local score) is convenient. As a result, the rate a convergence is actually $\sqrt{\ln(M)/M}$.

Table 2. Error factor.

Class	K	Mod	$\delta_{k,M}$ Min	$\delta_{k,M}$ Max	$R(M)$ Min	$R(M)$ Max
1	2	F	0.40	0.71	6.7	11.8
		V	0.42	0.71	5.6	6.7
	5	F	0.37	0.63	10.0	16.9
		V	0.37	0.63	9.0	10.0
	10	F	0.35	0.60	11.7	20.1
		V	0.37	0.60	10.0	11.6
2	2	F	0.45	0.70	6.0	9.4
		V	0.43	0.70	4.1	6.0
	5	F	0.36	0.61	8.1	13.6
		V	0.35	0.60	6.9	8.1
	10	F	0.33	0.56	9.3	15.7
		V	0.35	0.57	7.2	9.3
3	2	F	0.48	0.72	7.8	11.6
		V	0.48	0.73	5.3	7.8
	5	F	0.40	0.68	13.5	22.5
		V	0.42	0.67	11.6	13.5
	10	F	0.41	0.65	17.2	27.5
		V	0.41	0.65	15.2	17.3

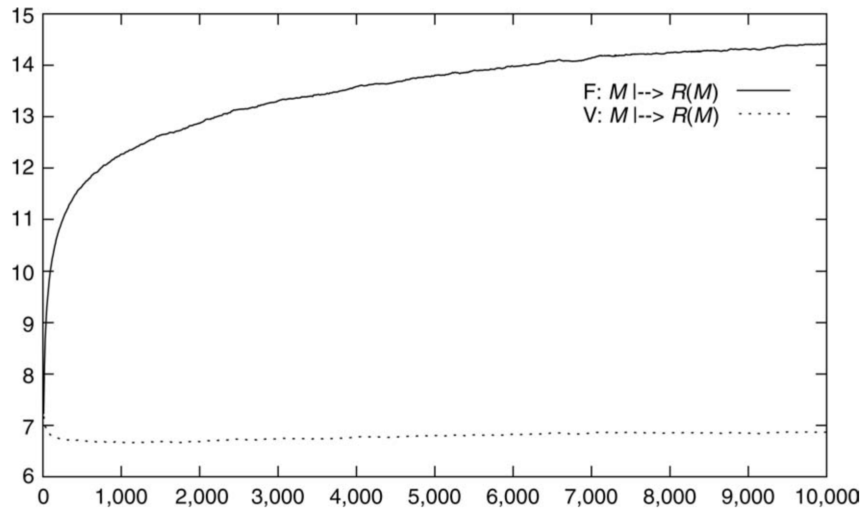

Figure 2. Graphs of $M \mapsto R(M)$, when $\mu = \mu_{2,2}$.

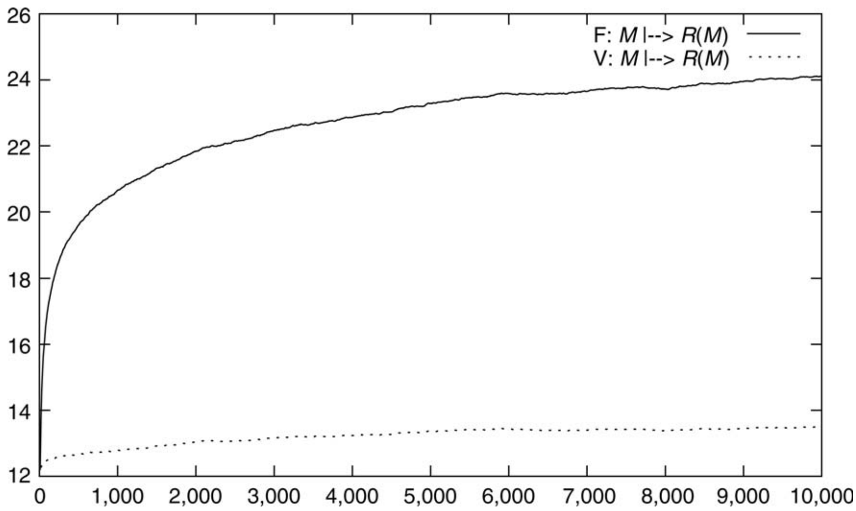

Figure 3. Graphs of $M \mapsto R(M)$, when $\mu = \mu_{3,2}$.

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